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Gibbs States, Algebraic Dynamics and Generalized Riesz Systems

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Abstract

In PT-quantum mechanics the generator of the dynamics of a physical system is not necessarily a self-adjoint Hamiltonian. It is now clear that this choice does not prevent to get a unitary time evolution and a real spectrum of the Hamiltonian, even if, most of the times, one is forced to deal with biorthogonal sets rather than with on orthonormal basis of eigenvectors. In this paper we consider some extended versions of the Heisenberg algebraic dynamics and we relate this analysis to some generalized version of Gibbs states and to their related KMS-like conditions. We also discuss some preliminary aspects of the Tomita–Takesaki theory in our context.

Keywords Gibbs states · Non-Hermitian Hamiltonians · Biorthogonal sets of vectors · Tomita–Takesaki theory

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1 Introduction

In the past 25 years or so it has become clearer and clearer that the role of self-adjointness of the observables of some given microscopic system can be, sometimes, relaxed, without modifying the essential benefits of dealing with, for instance, a self-adjoint Hamiltonian. In fact, we can still find real eigenvalues, a unitary time evolution and a preserved probability even if the requirement of the Hamiltonian being self-adjoint is replaced by some milder assumption, like in PT- or in pseudo-hermitian quantum mechanics. We refer to [1–5] for some references on these approaches, both from a more physical point of view and from their mathematical consequences.

Considering a non-selfadjoint Hamiltonian $H \neq H^*$ may lead to the appearance of new and often unpleasant features; for instance, the set $\{\varphi_n\}$ of eigenstates of H, if any, in general is no longer an orthonormal system, but this set $\{\varphi_n\}$ and the set $\{\psi_n\}$ of the eigenstates of H^* turn out to be biorthogonal i.e., $(\varphi_n|\psi_m) = \delta_{n,m}$. Also, in concrete examples they are not bases for the Hilbert space $\mathcal{H}$ where the model is defined, but they may still be complete in $\mathcal{H}$. This is the reason why the notion of $\mathcal{D}$-quasi bases was proposed in [6].

This concept can be thought as a suitable extension of Riesz biorthogonal bases, and similar biorthogonal sets are found in several concrete physical applications, playing often the role that in the traditional setup is played by orthonormal bases (ONB). In recent papers many other extensions of Riesz bases, mostly involving unbounded operators, have also been considered. In particular we mention generalized Riesz systems introduced by one of us (H.I) and analyzed in a series of papers [7–14]). For other studies on extensions of Riesz bases or on generalizations to different environments (Krein spaces, Rigged Hilbert spaces) we refer to [15–17].

In [18] the role of similar biorthogonal sets, in particular Riesz bases, in the analysis of Gibbs states, KMS condition and algebraic Heisenberg dynamics was first considered. More recently a similar analysis has been carried out by other authors (see, e.g. [19]). Here we want to give our contribution to this line of research, by using the biorthogonal sets originated by generalized Riesz systems.

The paper is organized as follows: in the next section we give some preliminaries. In Sect. 3 we propose our definition of Gibbs state defined by generalized Riesz systems, when the dynamics is driven by a self-adjoint operator H_0. The natural settings which we will adopt is the O^*-algebra $\mathcal{L}^\dagger(\mathcal{D})$, where $\mathcal{D}$ is a dense subspace of $\mathcal{H}$, [20–22]. This will appear to be a good choice, due to the fact that the operators appearing in our analysis are mostly unbounded. In Sect. 4 we will consider possible definitions of the algebraic dynamics for non self-adjoint Hamiltonians, and then we will consider how these dynamics are related to the generalized Gibbs states introduced first, and the KMS-like relations which arise from this construction. In Sect. 5 we will propose a preliminary analysis of the Tomita–Takesaki modular theory in our context, while our conclusions are given in Sect. 6.

2 Preliminaries

In this section we review the basic definitions and results on generalized Riesz systems and O^*-algebras needed in this paper.

Definition 2.1 A sequence $\{\varphi_n\}$ in a Hilbert space $\mathcal{H}$ with inner product $(\cdot|\cdot)$ is called a generalized Riesz system if there exist an ONB $\{f_n\}$ in $\mathcal{H}$ and a densely defined closed operator T in $\mathcal{H}$ with densely defined inverse, such that $\{f_n\} \subset D(T) \cap D((T^{-1})^*)$ and $Tf_n = \varphi_n$, $n = 0, 1, \ldots$. Such a $(\{f_n\}, T)$ is called a constructing pair for $\{\varphi_n\}$ and T is called a constructing operator for $\{\varphi_n\}$.

Suppose that $(\{\varphi_n\}, \{\psi_n\})$ is a biorthogonal pair such that $\{\varphi_n\}$ be a generalized Riesz system with a constructing pair $(\{f_n\}, T)$. Then putting $\psi_n^T = (T^{-1})^* f_n$, $n = 0, 1, \ldots$, $\{\varphi_n\}$ and $\{\psi_n^T\}$ are biorthogonal sequences, that is, $(\varphi_n|\psi_m^T) = \delta_{nm}$, $n, m = 0, 1, \ldots$. If $\psi_n^T = \psi_n$, $n = 0, 1, \ldots$, then $\{\psi_n\}$ is a generalized Riesz system with a constructing pair $(\{f_n\}, (T^{-1})^*)$. But, the equality $\psi_n^T = \psi_n$, $n = 0, 1, \ldots$ does not necessarily hold. If this equality holds, then T is called *natural* and $(\{f_n\}, T)$ is called *natural constructing pair*.

Let $\mathcal{D}$ be a dense subspace in $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all closable linear operators X in $\mathcal{H}$ such that $D(X) = \mathcal{D}$ and $D(X^*) \supset \mathcal{D}$. As usual we put, for $X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, $X^\dagger := X^* \lceil_{\mathcal{D}}$. Let

$$\mathcal{L}(\mathcal{D}) = \{X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H});\ X\mathcal{D} \subset \mathcal{D}\},$$
$$\mathcal{L}^\dagger(\mathcal{D}) = \{X \in \mathcal{L}(\mathcal{D});\ X^*\mathcal{D} \subset \mathcal{D}\}.$$

Then $\mathcal{L}(\mathcal{D})$ is an algebra with the usual operations: $X + Y$, αX and XY, and $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$-algebra with the involution $X \to X^\dagger := X^* \lceil_{\mathcal{D}}$, inherited by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. A $*$-subalgebra $\mathcal{M}$ of $\mathcal{L}^\dagger(\mathcal{D})$ is said to be an *O^*-algebra* on $\mathcal{D}$ in $\mathcal{H}$. Here we assume that $\mathcal{M}$ has the identity operator I. A locally convex topology defined by a family $\{\|\cdot\|_X;\ X \in \mathcal{M}\}$ of seminorms: $\|\xi\|_X := \|X\xi\|$, $\xi \in \mathcal{D}$ is called the *graph topology* on $\mathcal{D}$ and denoted by $t_{\mathcal{M}}$. If the locally convex space $\mathcal{D}[t_{\mathcal{M}}]$ is complete, then $\mathcal{M}$ is called *closed* and it is shown that $\mathcal{M}$ is closed if and only if $\mathcal{D} = \bigcap_{X\in\mathcal{M}} D(\bar{X})$. If $\mathcal{D} = \bigcap_{X\in\mathcal{M}} D(X^*)$, then $\mathcal{M}$ is called *self-adjoint*. Next we define a weak commutant of $\mathcal{M}$ as follows:

$$\mathcal{M}'_w := \{C \in B(\mathcal{H});\ (CX\xi|\eta) = (C\xi|X^\dagger\eta) \text{ for all } X \in \mathcal{M} \text{ and } \xi, \eta \in \mathcal{D}\},$$

where $B(\mathcal{H})$ is the C^*-algebra of all bounded linear operators on $\mathcal{H}$. Then $\mathcal{M}'_w$ is a weakly closed $*$-invariant subspace of $B(\mathcal{H})$, but it is not necessarily an algebra. If $\mathcal{M}$ is self-adjoint, then $\mathcal{M}'_w$ is a von Neumann algebra on $\mathcal{H}$ satisfying $\mathcal{M}'_w\mathcal{D} \subset \mathcal{D}$. Furthermore, we see that $\mathcal{L}^\dagger(\mathcal{D})'_w = \mathbb{C}I$. We define some topologies on $\mathcal{M}$. For any $\xi, \eta \in \mathcal{D}$ we put $p_{\xi,\eta}(X) := |(X\xi|\eta)|$, $p_\xi(X) := \|X\xi\|$, $X \in \mathcal{L}^\dagger(\mathcal{D})$. The locally convex topology on $\mathcal{L}^\dagger(\mathcal{D})$ defined by the family $\{p_{\xi,\eta}(\cdot);\ \xi, \eta \in \mathcal{D}\}$ (resp. $\{p_\xi(\cdot);\ \xi \in \mathcal{D}\}$) of seminorms on $\mathcal{L}^\dagger(\mathcal{D})$ is called the *weak* (resp. *strong*) topology, and the induced topology of the weak (resp. strong) topology on $\mathcal{M}$ is called the weak (resp. strong) topology on $\mathcal{M}$. For any $Y \in \mathcal{M}$ and $\xi \in \mathcal{D}$ we define a seminorm on

$\mathcal{M}$ by

$$p_{\xi,Y}(X) := \|YX\xi\|, \quad X \in \mathcal{M}.$$

The locally convex topology on $\mathcal{M}$ defined by the family $\{P_{\xi,Y}(\cdot);\ \xi \in \mathcal{D}, Y \in \mathcal{M}\}$ is called the *quasi-strong topology* on $\mathcal{M}$. A linear functional ω on $\mathcal{M}$ is called *positive* if $\omega(X^\dagger X) \geqq 0$ for all $X \in \mathcal{M}$, and a positive linear functional ω on $\mathcal{M}$ is a *state* if $\omega(I) = 1$. A $(*)$-isomorphism of $\mathcal{M}$ onto $\mathcal{M}$ is called a $(*)$-*automorphism* of $\mathcal{M}$ and $\{\alpha_t\}_{t\in\mathbb{R}}$ is called a *one-parameter group of* $(*)$-*automorphisms* of $\mathcal{M}$ if $\alpha_0(X) = X$ and $\alpha_{s+t}(X) = \alpha_s(\alpha_t(X))$ for all $X \in \mathcal{M}$. A one-parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ of automorphisms of $\mathcal{M}$ is *weakly* (resp. *strongly, quasi-strongly*) *continuous* if $\lim_{t\to 0}\alpha_t(X) = X$ for any $X \in \mathcal{M}$ under the weak (resp. strong, quasi-strong) topology. An operator H in $\mathcal{L}^\dagger(\mathcal{D})$ is called a *weak* (resp. *strong, quasi-strong*) *generator* for $\{\alpha_t\}_{t\in\mathbb{R}}$ if $\lim_{t\to 0}\frac{\alpha_t(X)-X}{t} = i[H, X]$ under the weak (resp. strong, quasi-strong) topology. For O^*-algebras refer to [23].

3 Gibbs States Defined by Generalized Riesz Systems

Throughout this section let $\{\varphi_n\}$ be a generalized Riesz system in a Hilbert space $\mathcal{H}$ with a constructing pair $(\{f_n\}, T)$ and $\lambda_n > 0,\ n = 0, 1, \ldots$, such that $\{e^{-\frac{\beta}{2}\lambda_n}\} \in \ell^1$, for some $\beta > 0$. In this section we shall define and investigate a Gibbs state ω_φ^β defined through $\{\varphi_n\}$ on the maximal O^*-algebra $\mathcal{L}^\dagger(\mathcal{D})$ on a dense subspace $\mathcal{D}$ in $\mathcal{H}$. We put

$$H_0 := \sum_{n=0}^{\infty} \lambda_n f_n \otimes \bar{f}_n,$$

where, for $x, y \in \mathcal{H}$, the operator $x \otimes \bar{y}$ is defined by

$$(x \otimes \bar{y})\xi = (\xi|y)x, \quad \xi \in \mathcal{H}.$$

Then H_0 is a non-singular positive self-adjoint operator in $\mathcal{H}$ such that

$$H_0 f_n = \lambda_n f_n, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

and it is called a standard hamiltonian for $\{f_n\}$.

Before entering in the main matter of the paper some comments are in order. Once H_0 and a generalized Riesz system $\{\varphi_n\}$ with constructing pair $(\{f_n\}, T)$ are given, one can define an operator H on the linear span D_φ of $\{\varphi_n\}$ by putting $H\varphi_n = \lambda_n\varphi_n;\ n \in \mathbb{N}_0$ and extending by linearity to D_φ. Since D_φ needs not be dense in $\mathcal{H}$, it is natural to consider H as an operator acting in $\mathcal{H}_\varphi$, the closure of D_φ in $\mathcal{H}$. It is then natural to write $H\varphi_n = HTf_n = \lambda_n\varphi_n = TH_0 f_n,\ n \in \mathbb{N}$, which looks like an *intertwining* (or, better, when T is invertible, a *similarity*) condition for H and $\mathcal{H}_0$, as discussed in [18] for Riesz bases. Similarity is a quite strong condition in particular when considering the spectrum of the involved operators or trying to get a functional calculus. We will not pursue this approach here because it doesn't fit with the general situation we are considering.

Lemma 3.1 *Let $\mathcal{D}$ be a dense subspace in $\mathcal{H}$. Suppose that*

$$e^{-\frac{\beta}{2}H_0}\mathcal{H} \subset \mathcal{D}. \tag{3.1}$$

Then $Xe^{-\beta H_0}$ is trace class on $\mathcal{H}$, for all $X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$.

Proof Take an arbitrary $X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Since $e^{-\frac{\beta}{2}H_0}\mathcal{H} \subset \mathcal{D}$, we have

$$D(Xe^{-\frac{\beta}{2}H_0}) = \mathcal{H}.$$

Thus, $Xe^{-\frac{\beta}{2}H_0}$ is an everywhere defined operator on $\mathcal{H}$ and it is simple to show that it is closable. Therefore $Xe^{-\frac{\beta}{2}H_0}$ is a closed operator in $\mathcal{H}$. By the closed graph theorem $Xe^{-\frac{\beta}{2}H_0}$ is a bounded operator on $\mathcal{H}$ and since

$$Xe^{-\beta H_0} = \left(Xe^{-\frac{\beta}{2}H_0}\right)e^{-\frac{\beta}{2}H_0},$$

we have $Xe^{-\beta H_0}$ is trace class. This completes the proof. □

We remark that any subspace $\mathcal{D}$ in $\mathcal{H}$ satisfying (3.1) is dense in $\mathcal{H}$ since it contains the ONB $\{f_n\}$ in $\mathcal{H}$, and $D^\infty(H_0) := \bigcap_{n\in\mathbb{N}} D(H_0^n)$ is a subspace in $\mathcal{H}$ satisfying (3.1).

Under assumption (3.1) we can introduce a state on $\mathcal{L}^\dagger(\mathcal{D})$ by

$$\omega_f^\beta(X) := \frac{1}{Z_f}\sum_{n=0}^{\infty} e^{-\beta\lambda_n}(Xf_n|f_n), \quad X \in \mathcal{L}^\dagger(\mathcal{D}),$$

where $Z_f := \sum_{n=0}^{\infty} e^{-\beta\lambda_n}$. Indeed, by Lemma 3.1 we have

$$\begin{aligned}\mathrm{tr}\,(Xe^{-\beta H_0}) &= \sum_{n=0}^{\infty}\left(Xe^{\beta H_0}f_n|f_n\right)\\ &= \sum_{n=0}^{\infty} e^{-\beta\lambda_n}(Xf_n|f_n),\end{aligned}$$

for all $X \in \mathcal{L}^\dagger(\mathcal{D})$, and hence ω_f^β is a state on $\mathcal{L}^\dagger(\mathcal{D})$, and it is called a *Gibbs state* on $\mathcal{L}^\dagger(\mathcal{D})$ for the ONB $\{f_n\}$. We formally define a Gibbs state ω_φ^β on $\mathcal{L}^\dagger(\mathcal{D})$ for the generalized Riesz system $\{\varphi_n\}$ by

$$\omega_\varphi^\beta(X) := \frac{1}{Z_\varphi}\sum_{n=0}^{\infty} e^{-\beta\lambda_n}(X\varphi_n|\varphi_n), \quad X \in \mathcal{L}^\dagger(\mathcal{D}),$$

where $Z_\varphi := \sum_{n=0}^{\infty} e^{-\beta\lambda_n} \|\varphi_n\|^2$. Conditions for that are discussed in [18]. In what follows we will consider only generalized Riesz system $\{\varphi_n\}$ for which $Z_\varphi < \infty$.

We do not know whether ω_φ^β is a state on $\mathcal{L}^\dagger(\mathcal{D})$, namely, in particular, $|\omega_\varphi^\beta(X)| < \infty$ for all $X \in \mathcal{L}^\dagger(\mathcal{D})$. For that, we assume that a constructing pair $(\{f_n\}, T)$ for a generalized Riesz system $\{\varphi_n\}$ satisfies the following

Assumption 1 There exists a dense subspace $\mathcal{D}$ in $\mathcal{H}$ satisfying

(i) $e^{-\frac{\beta}{2}H_0}\mathcal{H} \subset \mathcal{D}$,
(ii) $\mathcal{D} \subset D(T) \cap D(T^*)$,
(iii) $T\lceil_{\mathcal{D}}$ (the restriction of T to $\mathcal{D}$) $\in \mathcal{L}(\mathcal{D})$.

By (ii) in Assumption 1, $T\lceil_{\mathcal{D}} \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. In the rest of the paper we will use the same symbol for T, $e^{-\frac{\beta}{2}H_0}$ and $e^{-\beta H_0}$, and for their restrictions to $\mathcal{D}$. Then we have the following

Theorem 3.2 *Under Assumption 1, ω_φ^β is a faithful state on $\mathcal{L}^\dagger(\mathcal{D})$ and*

$$\begin{aligned}\omega_\varphi^\beta(X) &= \frac{1}{Z_\varphi} \operatorname{tr} \left(T^* X T e^{-\beta H_0}\right) \\ &= \frac{1}{Z_\varphi} \operatorname{tr} \left(\left(T e^{-\frac{\beta}{2}H_0}\right)^* X \left(T e^{-\frac{\beta}{2}H_0}\right)\right),\end{aligned}$$

for all $X \in \mathcal{L}^\dagger(\mathcal{D})$.

Here a state ω on $\mathcal{L}^\dagger(\mathcal{D})$ is said to be faithful if $\omega(X^\dagger X) = 0$, $X \in \mathcal{L}^\dagger(\mathcal{D})$, then $X = 0$.

Proof Take an arbitrary $X \in \mathcal{L}^\dagger(\mathcal{D})$. Then, by Assumption 1, $T^*XT \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ and by Lemma 3.1 $(T^*XT)e^{-\beta H_0}$ is trace class, which implies that

$$\begin{aligned}\frac{1}{Z_\varphi}\operatorname{tr} \left(T^* X T e^{-\beta H_0}\right) &= \frac{1}{Z_\varphi} \sum_{n=0}^{\infty} (T^* X T e^{-\beta H_0} f_n | f_n). \\ &= \frac{1}{Z_\varphi} \sum_{n=0}^{\infty} e^{-\beta\lambda_n} (T^* X T f_n | f_n) \\ &= \sum_{n=0}^{\infty} e^{-\beta\lambda_n} (X\varphi_n | \varphi_n),\end{aligned} \tag{3.2}$$

for all $X \in \mathcal{L}^\dagger(\mathcal{D})$. Hence $\omega_\varphi^\beta(X) = \frac{1}{Z_\varphi} \operatorname{tr} \left(T^* X T e^{-\beta H_0}\right)$ and it is a state on $\mathcal{L}^\dagger(\mathcal{D})$. Since T and T^* are non-singular; that is, T^{-1} and $(T^*)^{-1}$ exist, we see that ω_φ^β is faithful. Furthermore, we have

$$
\begin{aligned}
\omega_\varphi(X) &= \frac{1}{Z_\varphi}\,\mathrm{tr}\,\left(T^* X T e^{-\beta H_0}\right) \\
&= \frac{1}{Z_\varphi}\,\mathrm{tr}\,\left(\left(T^* X T e^{-\frac{\beta}{2}H_0}\right) e^{-\frac{\beta}{2}H_0}\right) \\
&= \frac{1}{Z_\varphi}\,\mathrm{tr}\,\left(e^{-\frac{\beta}{2}H_0}(T^* X T) e^{-\frac{\beta}{2}H_0}\right) \\
&= \frac{1}{Z_\varphi}\,\mathrm{tr}\,\left(\left(T e^{-\frac{\beta}{2}H_0}\right)^* X \left(T e^{-\frac{\beta}{2}H_0}\right)\right),
\end{aligned}
$$

for all $X \in \mathcal{L}^\dagger(\mathcal{D})$. This completes the proof. □

Remark Clearly, $\omega_\varphi(X) = \frac{Z_0}{Z_\varphi}\omega_0(T^* X T)$, for every $X \in \mathcal{L}^\dagger(\mathcal{D})$, where $\omega_0(X) = \frac{1}{Z_0}\mathrm{tr}(X e^{-\beta H_0})$ and $Z_0 = \sum_{n=0}^{\infty} e^{-\beta\lambda_n}$, as usually introduced in the literature when in presence of a self-adjoint Hamiltonian H_0.

Now we put

$$
\psi_n := (T^{-1})^* f_n, \quad n \in \mathbb{N}_0.
$$

Then $\{\psi_n\}$ is a generalized Riesz system with a constructing pair $(\{f_n\}, (T^{-1})^*)$ and $\{\varphi_n\}$ and $\{\psi_n\}$ are biorthogonal sequences. For the constructing operator $(T^{-1})^*$ for $\{\psi_n\}$, we assume the following, which is completely analogous to what stated in Assumption 1 above.

Assumption 2 Assume that there exists a dense subspace $\mathcal{E}$ in $\mathcal{H}$ satisfying

(i) $e^{-\frac{\beta}{2}H_0}\mathcal{H} \subset \mathcal{E}$,
(ii) $\mathcal{E} \subset D(T^{-1}) \cap D((T^{-1})^*)$,
(iii) $(T^{-1})^* \lceil_{\mathcal{E}} \in \mathcal{L}(\mathcal{E})$.

As before, we use the same symbol for the operators $(T^{-1})^*$, $e^{-\frac{\beta}{2}H_0}$ and $e^{-\beta H_0}$ and for their restrictions to $\mathcal{E}$. Now we define put

$$
\omega_\psi^\beta(X) := \frac{1}{Z_\psi}\sum_{n=0}^{\infty} e^{-\beta\lambda_n}(\psi_n|\psi_n), \quad X \in \mathcal{L}^\dagger(\mathcal{E}),
$$

where $Z_\psi := \sum_{n=0}^{\infty} e^{-\beta\lambda_n}\|\psi_n\|^2$, which is assumed to exist finite, see [18]. Then we have the following

Theorem 3.3 *Under Assumption 2, ω_ψ^β is a faithful state on $\mathcal{L}^\dagger(\mathcal{E})$ and*

$$
\begin{aligned}
\omega_\psi(X) &= \frac{1}{Z_\psi}\,\mathrm{tr}\,\left(T^{-1} X (T^{-1})^* e^{-\beta H_0}\right) \\
&= \frac{1}{Z_\psi}\,\mathrm{tr}\,\left(\left((T^{-1})^* e^{-\frac{\beta}{2}H_0}\right)^* X (T^{-1})^* e^{-\frac{\beta}{2}H_0}\right),
\end{aligned}
$$

for all $X \in \mathcal{L}^\dagger(\mathcal{E})$.

Proof It is proved similarly to Theorem 3.2. □

By Theorems 3.2 and 3.3 we have the following

Corollary 3.4 *Let $\{\varphi_n\}$ and $\{\psi_n\}$ be biorhotogonal sequences and $\{\varphi_n\}$ be generalized Riesz system with natural constructing pair $(\{f_n\}, T)$. Suppose that Assumptions 1 and 2 are satisfied, with $\mathcal{D} = \mathcal{E}$. Then the Gibbs states ω_φ^β and ω_ψ^β are faithful states on $\mathcal{L}^\dagger(\mathcal{D})$ satisfying*

$$\begin{aligned}\omega_\varphi^\beta(X) &= \frac{1}{Z_\varphi}\ \mathrm{tr}\ \left((T^*XT)e^{-\beta H_0}\right)\\ &= \frac{1}{Z_\varphi}\ \mathrm{tr}\ \left(\left(Te^{-\frac{\beta}{2}H_0}\right)^* X\left(Te^{-\frac{\beta}{2}H_0}\right)\right)\end{aligned}$$

and

$$\begin{aligned}\omega_\psi^\beta(X) &= \frac{1}{Z_\psi}\ \mathrm{tr}\ \left(T^{-1}X(T^{-1})^*e^{-\beta H_0}\right)\\ &= \frac{1}{Z_\psi}\ \mathrm{tr}\ \left(\left((T^{-1})^*e^{-\frac{\beta}{2}H_0}\right)^* X\left((T^{-1})^*e^{-\frac{\beta}{2}H_0}\right)\right),\end{aligned}$$

for all $X \in \mathcal{L}^\dagger(\mathcal{D})$.

Corollary 3.5 *Let $\{\varphi_n\}$ be a Riesz basis with a constructing pair $(\{f_n\}, T)$. Suppose that there exists a dense subspace $\mathcal{D}$ in $\mathcal{H}$ such that*

(i) $e^{-\frac{\beta}{2}H_0}\mathcal{H} \subset \mathcal{D}$.
(ii) $T\mathcal{D} \subset \mathcal{D}$.
(iii) $(T^{-1})^*\mathcal{D} \subset \mathcal{D}$.

Then the Gibbs states ω_φ^β and ω_ψ^β are faithful states on $\mathcal{L}^\dagger(\mathcal{D})$.

4 Dynamics and KMS-Like Condition

4.1 Standard Heisenberg Time Evolution

Let H_0 be a non-singular positive self-adjoint operator in $\mathcal{H}$ satisfying $H_0 = \sum_{n=0}^\infty \lambda_n f_n \otimes f_n$, where $\{f_n\}$ is an ONB in a Hilbert space $\mathcal{H}$ and $\{\lambda_n\}$ is a sequence of strictly positive numbers satisfying $\sum_{n=0}^\infty e^{-\frac{1}{2}\lambda_n} < \infty$, and $\mathcal{D}$ be a dense subspace in $\mathcal{H}$ such that

$$H_0\mathcal{D} \subset \mathcal{D} \quad \text{and} \quad e^{itH_0}\mathcal{D} \subset \mathcal{D} \quad \text{for all} \quad t \in \mathbb{R}. \tag{4.1}$$

For example, $\mathcal{D} = D^\infty(H_0) := \cap_{n\in\mathbb{N}_0} D(H_0^n)$ satisfies (4.1). Indeed, since $H_0^n e^{itH_0}x = e^{itH_0}H_0^n x$, $x \in D^\infty(H_0)$ we have $e^{itH_0}x \in D(H_0^n)$, for all $n \in \mathbb{N}_0$. Here we put

$$\alpha_t^0(X) := e^{itH_0}Xe^{-itH_0}, \quad X \in \mathcal{L}^\dagger(\mathcal{D}),\ t \in \mathbb{R}.$$

Then, $\{\alpha_t^0\}_{t\in\mathbb{R}}$ is a one-parameter group of $*$-automorphisms of $\mathcal{L}^\dagger(\mathcal{D})$, and we have the following

Lemma 4.1.1 *Suppose that* (4.1) *is satisfied and that the one-parameter unitary group* $\{e^{itH_0}\}_{t\in\mathbb{R}}$ *is quasi-strongly continuous on* $\mathcal{L}^\dagger(\mathcal{D})$.
Then, $\{\alpha_t^0\}_{t\in\mathbb{R}}$ *is strongly continuous and its weak generator is* H_0. *In particular, if* $\mathcal{D} = D^\infty(H_0)$, *then* $\{\alpha_t^0\}_{t\in\mathbb{R}}$ *is a quasi-strongly continuous one-parameter group of* $*$*-automorphisms of* $\mathcal{L}^\dagger(D^\infty(H_0))$ *and its quasi-strong generator is* H_0.

Proof First, we show that $\{\alpha_t^0\}$ is strongly continuous. Take arbitrary $X \in \mathcal{L}^\dagger(\mathcal{D})$ and $\xi \in \mathcal{D}$. Then by assumption we have

$$\begin{aligned}
\lim_{t\to 0} \|\alpha_t^0(X)\xi - X\xi\| &= \lim_{t\to 0} \|e^{itH_0}Xe^{-itH_0}\xi - X\xi\| \\
&\leqq \lim_{t\to 0} \left\{\|e^{itH_0}Xe^{-itH_0}\xi - e^{itH_0}X\xi\| + \|e^{itH_0}X\xi - X\xi\|\right\} \\
&\leqq \lim_{t\to 0} \|Xe^{-itH_0}\xi - X\xi\| + \lim_{t\to 0} \|e^{itH_0}X\xi - X\xi\| \\
&= 0.
\end{aligned}$$

Thus, $\{\alpha_t^0\}_{t\in\mathbb{R}}$ is strongly continuous.

Next, we show that H_0 is a weak generator of $\{\alpha_t^0\}_{t\in\mathbb{R}}$. Take arbitrary $X \in \mathcal{L}^\dagger(\mathcal{D})$ and $\xi, \eta \in \mathcal{D}$. Then it follows from our assumptions that

$$\begin{aligned}
\left(\frac{\alpha_0^t(X) - X}{t}\xi|\eta\right) &= \left(\frac{e^{itH_0}Xe^{-itH_0}\xi - e^{itH_0}X\xi + e^{itH_0}X\xi - X\xi}{t}|\eta\right) \\
&= \left(\frac{e^{-itH_0}\xi - \xi}{t}|X^\dagger e^{-itH_0}\eta\right) + \left(\frac{e^{tH_0}X\xi - X\xi}{t}|\eta\right) \\
&\to (-iH_0\xi|X^\dagger\eta) + (iH_0X\xi|\eta) \quad \text{as} \quad t \to 0 \\
&= (i[H_0, X]\xi|\eta),
\end{aligned}$$

which yields that H_0 is a weak generator of $\{\alpha_t^0\}_{t\in\mathbb{R}}$.

Let $\mathcal{D} = D^\infty(H_0)$ and t_{H_0} be a locally convex topology on $\mathcal{D}$ defined by a sequence $\{\|\cdot\|_{H_0^n};\ n \in \mathbb{N}_0\}$ of norms on $\mathcal{D}$. Since $H_0^n \in \mathcal{L}^\dagger(\mathcal{D})$, for all $n \in \mathbb{N}$, we have $t_{H_0} \prec t_{\mathcal{L}^\dagger(\mathcal{D})}$. Conversely we show that $t_{\mathcal{L}^\dagger(\mathcal{D})} \prec t_{H_0}$. Take an arbitrary $X \in \mathcal{L}^\dagger(\mathcal{D})$. Since the identity ι is a closed map of the Fréchet space $\mathcal{D}[t_{H_0}]$ into the Hilbert space $\mathcal{D}(\|\cdot\|_{\bar{X}})$ with the graph norm $\|\cdot\|_{\bar{X}} := \|\cdot\| + \|\bar{X}\cdot\|$, it follows from the closed graph theorem that it is continuous, which implies that $t_{\mathcal{L}^\dagger(\mathcal{D})} \prec t_{H_0}$. Thus we have

$$t_{H_0} = t_{\mathcal{L}^\dagger(\mathcal{D})} \tag{4.2}$$

and for any $X \in \mathcal{L}^\dagger(\mathcal{D})$ there exist $n \in \mathbb{N}$ and $\gamma > 0$ such that

$$\|X\xi\| \leqq \gamma\|\xi\|_{H_0^n} \quad \text{for all} \quad \xi \in \mathcal{D}. \tag{4.3}$$

Then, for any $X, Y \in \mathcal{L}^\dagger(\mathcal{D})$ and $\xi \in \mathcal{D}$ it follows from $H_0^n X \in \mathcal{L}^\dagger(\mathcal{D})$ and by our assumptions that

$$\begin{aligned}
&\|Y\alpha_t^0(X)\xi - YX\xi\| \\
&\leqq \gamma \|H_0^n \alpha_t^0(X)\xi - H_0^n X\xi\| \\
&= \gamma \|H_0^n e^{itH_0} X e^{-itH_0}\xi - H_0^n X\xi\| \\
&\leqq \gamma \left\{\|H_0^n e^{itH_0} X e^{-itH_0}\xi - e^{itH_0} H_0^n X\xi\| + \|e^{itH_0} H_0^n X\xi - H_0^n X\xi\|\right\} \\
&= \gamma \left\{\|H_0^n X(e^{-itH_0}\xi - \xi)\| + \|e^{itH_0} H_0^n X\xi - H_0^n X\xi\|\right\} \\
&\quad \to 0 \quad \text{as} \quad t \to 0,
\end{aligned}$$

which implies that α_t^0 is quasi-strongly continuous. Furthermore, we have

$$\begin{aligned}
&H_0^m \left(\frac{\alpha_0^t(X)\xi - X\xi}{t} - i[H_0, X]\xi\right) \\
&= H_0^m \left(\frac{e^{itH_0} X e^{-itH_0}\xi - e^{itH_0} X\xi + e^{itH_0} X\xi - X\xi}{t} - i[H_0, X]\xi\right) \\
&= H_0^m \left\{\left(e^{itH_0} \frac{Xe^{-itH_0}\xi - X\xi}{t} + iXH_0\xi\right) + \left(\frac{e^{itH_0} X\xi - X\xi}{t} - iH_0X\xi\right)\right\} \\
&= \left(e^{itH_0} H_0^m \frac{Xe^{-itH_0}\xi - X\xi}{t} + iH_0^m XH_0\xi\right) + H_0^m \left(\frac{e^{itH_0} X\xi - X\xi}{t} - iH_0X\xi\right). \\
&= e^{itH_0} \left(H_0^m \frac{Xe^{-itH_0}\xi - X\xi}{t} + iH_0^m XH_0\xi\right) - iH_0^m \left(e^{itH_0} XH_0\xi - XH_0\xi\right) \\
&\quad + H_0^m \left(\frac{e^{itH_0} X\xi - X\xi}{t} - iH_0X\xi\right).
\end{aligned} \tag{4.4}$$

Then, it follows from $H_0^m X \in \mathcal{L}^\dagger(\mathcal{D})$ and (4.3) that

$$\begin{aligned}
\left\|H_0^m X \left(\frac{e^{-itH_0}\xi - \xi}{t} + iH_0\xi\right)\right\| &\leqq \gamma' \left\|H_0^{n'} \left(\frac{e^{-itH_0}\xi - \xi}{t} + iH_0\xi\right)\right\| \\
&= \gamma' \left\|\frac{e^{-itH_0} H_0^{n'}\xi - H_0^{n'}\xi}{t} + iH_0 H_0^{n'}\xi\right\| \\
&\to \gamma' \left\|-iH_0 H_0^{n'}\xi + iH_0 H_0^{n'}\xi\right\| = 0 \quad \text{as} \quad t \to 0
\end{aligned}$$

and from (ii) that

$$H_0^m \left(e^{itH_0} XH_0\xi - XH_0\xi\right) \to 0 \quad \text{as} \quad t \to 0,$$

which implies by (4.2) and (4.4) that $\lim_{t\to 0} \frac{\alpha_t^0(X) - X}{t} = i[H_0, X]$ under the quasi-strong topology. This completes the proof. □

4.2 The Heisenberg Time Evolution for Generalized Riesz Systems

Let $\{\varphi_n\}$ be a generalized Riesz system with a constructing pair $(\{f_n\}, T)$. We assume the following

Assumption 3 There exists a dense subspace $\mathcal{D}$ in $\mathcal{H}$ such that

(i) $\{f_n\} \subset \mathcal{D}$, for all $n \in \mathbb{N}_0$,
(ii) $H_0\mathcal{D} \subset \mathcal{D}$ and $e^{itH_0}\mathcal{D} \subset \mathcal{D}$, for all $t \in \mathbb{R}$,
(iii) $T\mathcal{D} = \mathcal{D}$ and $T^*\mathcal{D} = \mathcal{D}$,
(iv) $\{e^{itH_0}\}_{t\in\mathbb{R}}$ is quasi-strongly continuous.

Henceforth we denote an operator $A\lceil_{\mathcal{D}} \in \mathcal{L}^\dagger(\mathcal{D})$ by A for simplicity. Then, we have φ_n, $\psi_n := (T^\dagger)^{-1} f_n \in \mathcal{D}$, for all $n \in \mathbb{N}_0$, and we can define a non-self-adjoint operator H by $H := TH_0T^{-1}$. Then $H \in \mathcal{L}^\dagger(\mathcal{D})$ with $H^\dagger = (T^\dagger)^{-1}H_0T^\dagger$ and $H\varphi_n = \lambda_n\varphi_n$ and $H^\dagger\psi_n = \lambda_n\psi_n$, $n \in \mathbb{N}_0$ (we notice that (iii) implies that $(T^\dagger)^{-1} = (T^*)^{-1}\lceil_{\mathcal{D}}$ and then $(T^\dagger)^{-1} = (T^{-1})^\dagger$). Hence H and $H^\dagger$ can be considered as non-self-adjoint hamiltonians for $\{\varphi_n\}$ and $\{\psi_n\}$, respectively. Furthermore, take arbitrary $\xi, \eta \in \mathcal{D}$ and $t \in \mathbb{R}$. By Assumption 3, (iii) there exists a element $\zeta \in \mathcal{D}$ such that $\xi = T\zeta$. Then it follows that

$$\begin{aligned}
\left(\left(\sum_{k=0}^{n} \frac{1}{k!}(it)^k H^k\right)\xi|\eta\right) &= \left(T\left(\sum_{k=0}^{n} \frac{1}{k!}(it)^k H_0^k\right)T^{-1}T\zeta|\eta\right)\\
&= \left(\sum_{k=0}^{n} \frac{1}{k!}(it)^k H_0^k\zeta|T^\dagger\eta\right)\\
&\to \left(e^{itH_0}\zeta|T^\dagger\eta\right) \quad \text{as} \quad n \to \infty\\
&= \left(Te^{itH_0}T^{-1}\xi|\eta\right).
\end{aligned}$$

Hence, $T(\sum_{k=0}^{n} \frac{1}{k!}(it)^k H_0^k)T^{-1}$ converges weakly to $Te^{itH_0}T^{-1}$ on $\mathcal{D}$.

Similarly, $(T^\dagger)^{-1}(\sum_{k=0}^{n} \frac{1}{k!}(it)^k H_0^k)T^\dagger$ converges weakly to $(T^\dagger)^{-1}e^{itH_0}T^\dagger$ on $\mathcal{D}$. Thus, it is natural to define e^{itH} and $e^{itH^\dagger}$ by

$$e^{itH} := Te^{itH_0}T^{-1} \quad \text{and} \quad e^{itH^\dagger} := (T^\dagger)^{-1}e^{itH_0}T^\dagger \quad t \in \mathbb{R}. \tag{4.5}$$

Then we have the following

Lemma 4.2.1 *$\{e^{itH}\}_{t\in\mathbb{R}}$ and $\{e^{itH^\dagger}\}_{t\in\mathbb{R}}$ are quasi-strongly continuous one-parameter groups of $\mathcal{L}^\dagger(\mathcal{D})$ satisfying $(e^{itH})^\dagger = e^{-itH^\dagger}$, for all $t \in \mathbb{R}$.*

Proof By (4.5) it is immediately shown that $\{e^{itH}\}$ and $\{e^{itH^\dagger}\}$ are one-parameter groups of $\mathcal{L}^\dagger(\mathcal{D})$ satisfying $(e^{itH})^\dagger = e^{-itH^\dagger}$, for all $t \in \mathbb{R}$. We show that they are quasi-strongly continuous. Indeed, it follows from Assumption 3, (iv) that for any $X \in \mathcal{L}^\dagger(\mathcal{D})$ and $\xi \in \mathcal{D}$

$$\begin{aligned}\lim_{t\to 0}\|Xe^{itH}\xi - X\xi\| &= \lim_{t\to 0}\|XTe^{itH_0}T^{-1}\xi - X\xi\| \\ &= \lim_{t\to 0}\|XT(e^{itH_0}\eta - \eta)\| \\ &= 0,\end{aligned}$$

where $\eta \in \mathcal{D}$ with $T\eta = \xi$.

Similarly, we can show that $\{e^{itH^\dagger}\}$ is quasi-strongly continuous. □

We now define what we call the *Heisenberg time evolution for* $\{\varphi_n\}$ and $\{\psi_n\}$ as follows:

$$\alpha_t^\varphi(X) := e^{itH}Xe^{-itH} \quad \text{and} \quad \alpha_t^\psi(X) := e^{itH^\dagger}Xe^{-itH^\dagger}, \quad X \in \mathcal{L}^\dagger(\mathcal{D}),\ t \in \mathbb{R}.$$

By (4.5) we see that

$$\alpha_t^\varphi(X) = e^{itH}Xe^{-itH} = Te^{itH_0}T^{-1}XTe^{-itH_0}T^{-1} = T\alpha_t^0(T^{-1}XT)T^{-1},$$

where α_t^0 was defined before. This is in complete agreement with what originally proposed in [18]. Analogously,

$$\alpha_t^\psi(X) = (T^\dagger)^{-1}\alpha_t^0(T^\dagger X(T^\dagger)^{-1})T^\dagger.$$

Then we have the following

Theorem 4.2.2 $\{\alpha_t^\varphi\}_{t\in\mathbb{R}}$ *and* $\{\alpha_t^\psi\}_{t\in\mathbb{R}}$ *are weakly continuous one-parameter groups of automorphisms of* $\mathcal{L}^\dagger(\mathcal{D})$ *satisfying* $\alpha_t^\varphi(X)^\dagger = \alpha_t^\psi(X^\dagger)$*, for all* $X \in \mathcal{L}^\dagger(\mathcal{D})$ *and* $t \in \mathbb{R}$*. Furthermore their weak generators are* H *and* $H^\dagger$*, respectively. Moreover, in particular, if* $T \in B(\mathcal{H})$ *(resp.* $T^{-1} \in B(\mathcal{H})$*), then* $\{\alpha_t^\varphi\}$ *(resp.* $\{\alpha_t^\psi\}$*) is strongly continuous.*

Proof By Lemma 4.2.1, $\{\alpha_t^\varphi\}_{t\in\mathbb{R}}$ and $\{\alpha_t^\psi\}_{t\in\mathbb{R}}$ are one-parameter groups of automorphisms of $\mathcal{L}^\dagger(\mathcal{D})$ satisfying $\alpha_t^\varphi(X)^\dagger = \alpha_t^\psi(X^\dagger)$, for all $X \in \mathcal{L}^\dagger(\mathcal{D})$ and $t \in \mathbb{R}$. Let us now show that $\{\alpha_t^\varphi\}_{t\in\mathbb{R}}$ and $\{\alpha_t^\psi\}_{t\in\mathbb{R}}$ are weakly continuous. Take arbitrary $X \in \mathcal{L}^\dagger(\mathcal{D})$ and $\xi, \eta \in \mathcal{D}$. Since $\alpha_t^\varphi(X) = T\alpha_t^0(T^{-1}XT)T^{-1}$, for all $t \in \mathbb{R}$, it follows from Lemma 4.1.1 that

$$\begin{aligned}\left(\alpha_t^\varphi(X)\xi|\eta\right) &= \left(T\alpha_t^0(T^{-1}XT)T^{-1}T\zeta|\eta\right) \\ &= \left(\alpha_t^0(T^{-1}XT)\zeta|T^\dagger\eta\right) \\ &\to \left(T^{-1}XT\zeta|T^\dagger\eta\right) \quad \text{as } t \to 0 \\ &= (X\xi|\eta),\end{aligned}$$

which yields that $\{\alpha_t^\varphi\}_{t\in\mathbb{R}}$ is weakly continuous. Next we show that H is a weak generator of $\{\alpha_t^\varphi\}_{t\in\mathbb{R}}$. By Lemma 4.1.1, it follows that

$$\begin{aligned}\left(\left(\frac{\alpha_t^\varphi(X)-X}{t}\right)\xi|\eta\right) &= \left(\frac{T\alpha_t^0(T^{-1}XT)T^{-1}-X}{t}T\zeta|\eta\right)\\ &= \left(\frac{\alpha_t^0(T^{-1}XT)-T^{-1}XT}{t}\zeta|T^\dagger\eta\right)\\ &\to \left(i[H_0,T^{-1}XT]\zeta|T^\dagger\eta\right) \quad \text{as} \quad t\to 0\\ &= i\left(T(H_0T^{-1}XT-T^{-1}XTH_0)\zeta|\eta\right)\\ &= i\,((HX-XH)T\zeta|\eta)\\ &= i\,([H,X]\xi|\eta)\,.\end{aligned}$$

Thus, H is a weak generator of $\{\alpha_t^\varphi\}_{t\in\mathbb{R}}$. Similarly we can show that $\{\alpha_t^\psi\}_{t\in\mathbb{R}}$ is weakly continuous and its weak generator is $H^\dagger$. Finally, we show that if $T\in B(\mathcal{H})$, then $\{\alpha_t^\varphi\}_{t\in\mathbb{R}}$ is strongly continuous. Take arbitrary $X\in\mathcal{L}^\dagger(\mathcal{D})$ and $\xi\in\mathcal{D}$. Then, as usual, there exists an element $\zeta\in\mathcal{D}$ such that $\xi=T\zeta$ and by Assumption 3, (iii) we have

$$\begin{aligned}\|\alpha_t^\varphi(X)\xi-X\xi\| &= \|Te^{itH_0}T^{-1}XTe^{-itH_0}T^{-1}T\zeta-T^{-1}TXT\zeta\|\\ &\leqq \|Te^{itH_0}T^{-1}XTe^{-itH_0}\zeta-Te^{itH_0}T^{-1}XT\zeta\|\\ &\quad+\|Te^{itH_0}T^{-1}XT\zeta-TT^{-1}XT\zeta\|\\ &\leqq \|T\|\|T^{-1}XTe^{-itH_0}\zeta-T^{-1}XT\zeta\|\\ &\quad+\|T\|\|e^{itH_0}T^{-1}XT\zeta-T^{-1}XT\zeta\|\\ &\quad\to 0 \quad \text{as} \quad t\to 0.\end{aligned}$$

Similarly, if $T^{-1}\in B(\mathcal{H})$, then we can show that $\{\alpha_t^\psi\}_{t\in\mathbb{R}}$ is strongly continuous. This completes the proof. □

Next, let us consider the case of $\mathcal{D}=D^\infty(H_0)$. Then, Assumption 3, (i) and (ii) hold automatically, and (iv) holds from (4.1.2). Therefore, the following result easily follows.

Corollary 4.2.3 *Suppose that*

$$TD^\infty(H_0)=D^\infty(H_0) \quad \text{and} \quad T^*D^\infty(H_0)=D^\infty(H_0).$$

Then $\{\alpha_t^\varphi\}_{t\in\mathbb{R}}$ and $\{\alpha_t^\psi\}_{t\in\mathbb{R}}$ are quasi-strongly continuous and their quasi strong generators are H and $H^\dagger$, respectively.

Proof Since $t_{\mathcal{L}^\dagger(\mathcal{D})}=t_{H_0}$ by (4.2), for any $X\in\mathcal{L}^\dagger(\mathcal{D})$ there exist $n\in\mathbb{N}$ and $r>0$ such that

$$\|X\xi\|\leqq r\|\xi\|_{H_0^n} \quad \text{for all} \quad \xi\in\mathcal{D}. \tag{4.6}$$

For any $X, Y \in \mathcal{L}^\dagger(\mathcal{D})$ and $\xi \in \mathcal{D}$ with $\xi = T\zeta$ for some $\zeta \in \mathcal{D}$, it follows from (4.6) and Assumption 3, (iv) that

$$\begin{aligned}
\|Y\alpha_t^\varphi(X)\xi - YX\xi\| &= \|YTe^{itH_0}T^{-1}XTe^{-itH_0}T^{-1}T\zeta - YTT^{-1}XT\zeta\| \\
&\leqq \|YT(e^{itH_0}T^{-1}XTe^{-itH_0}\zeta - e^{itH_0}T^{-1}XT\zeta)\| \\
&\quad + \|YT(e^{itH_0}T^{-1}XT\zeta - T^{-1}XT\zeta)\| \\
&\leqq r_1\{\|H_0^{n_1}e^{itH_0}\left(T^{-1}XTe^{-itH_0}\eta - T^{-1}XT\eta\right)\| \\
&\quad + \|H_0^{n_1}(e^{itH_0}T^{-1}XT\eta - T^{-1}XT)\eta\|\} \\
&= r_1\{\|H_0^{n_1}T^{-1}XT(e^{-itH_0}\zeta - \zeta)\| \\
&\quad + \|H_0^{n_1}(e^{itH_0}T^{-1}XT\zeta - T^{-1}XT)\zeta\|\} \\
&\leqq r_1r_2\|H_0^{n_2}(e^{-itH_0}\zeta - \zeta)\| \\
&\quad + r_1\|H_0^{n_1}(e^{itH_0}T^{-1}XT\zeta - T^{-1}XT\zeta)\| \\
&= r_1r_2\|(e^{-itH_0} - I)H_0^{n_2}\zeta\| \\
&\quad + r_1\|(e^{itH_0} - I)(H_0^{n_1}T^{-1}XT\zeta)\| \\
&\quad \to 0 \quad \text{as} \quad t \to 0.
\end{aligned}$$

Thus $\{\alpha_t^\varphi\}$ is quasi-strongly continuous. We show that the quasi-strong generator of $\{\alpha_t^\varphi\}$ equals H. Indeed, take arbitrary $X, Y \in \mathcal{L}^\dagger(\mathcal{D})$ and $\xi \in \mathcal{D}$. Then $\xi = T\zeta$ for some $\zeta \in \mathcal{D}$ and by Lemma 4.1.1 the generator of $\{\alpha_t^0\}$ equals H_0, which yields that

$$\begin{aligned}
&\left\| Y\left(\frac{\alpha_t^\varphi(X) - X}{t}\right)\xi - Y(i[H, X])\xi \right\| \\
&= \left\| Y\left(\frac{T\alpha_t^0(T^{-1}XT)T^{-1} - X}{t}\right)T\zeta - iYT[H_0, T^{-1}XT]\zeta \right\| \\
&= \left\| YT\left(\frac{\alpha_t^0(T^{-1}XT) - T^{-1}XT}{t}\zeta - i[H_0, T^{-1}XT]\zeta\right) \right\| \\
&\to 0 \quad \text{as} \quad t \to 0.
\end{aligned}$$

Thus, the quasi-strong generator of $\{\alpha_t^\varphi\}$is H. Similarly, we can show that $\{\alpha_t^\psi\}$ is quasi-strongly continuous and its quasi-strong generator of $\{\alpha_t^\psi\}$ is $H^\dagger$. This completes the proof. □

4.3 Few Words on Generalized Von Neumann Entropy

In this section we briefly show how what is done with the dynamics can be repeated for the von Neumann entropy. We work here under a slightly generalized version of Assumption 3. In particular, we assume (i) and (iii) hold as in Assumption 3, and that t in (ii) can be complex-valued, $t = t_r + it_i$, with $t_i > 0$. More explicitly we assume that $e^{itH_0}\mathcal{D} \subset \mathcal{D}$, for all $t \in \mathbb{C}$, with Im $t > 0$. Assumption (3.iv) is not relevant for us here, and will not be considered. Our original assumption on the eigenvalues λ_n,

$\sum_{n=0}^{\infty} e^{-\frac{1}{2}\lambda_n} < \infty$, is here replaced by the stronger assumptions

$$\sum_{n=0}^{\infty} e^{-\gamma\lambda_n} < \infty, \qquad \sum_{n=0}^{\infty} \lambda_n e^{-\gamma\lambda_n} < \infty, \tag{4.7}$$

for all $\gamma > 0$. Therefore, in particular we have $Z_0(\beta) = \sum_{n=0}^{\infty} e^{-\beta\lambda_n} < \infty$. To simplify our treatment, from now on we will assume the following normalization: $Z_0(\beta) = 1$. Here β is just a positive parameter which, in the following section, will acquire an explicit physical meaning, the inverse temperature of a given system.

The von Neumann entropy connected to the self-adjoint Hamiltonian H_0 is defined as

$$S_{\rho_0} = -\mathrm{tr}\,(\rho_0 \log \rho_0)\,,$$

where, with our normalization, $\rho_0 = e^{-\beta H_0}$. A straightforward computation of S_{ρ_0} produces $S_{\rho_0} = \beta \sum_{n=0}^{\infty} \lambda_n e^{-\gamma\lambda_n}$, which is finite because of our assumption (4.7).

With the same steps as in the definition of e^{itH} and $e^{itH^\dagger}$, using our stronger assumptions, we conclude that $\sum_{k=0}^{n} \frac{1}{k!}(-\beta)^k H^k$ converges weakly to $Te^{-\beta H_0}T^{-1} = T\rho_0 T^{-1}$ on $\mathcal{D}$, and $\sum_{k=0}^{n} \frac{1}{k!}(-\beta)^k {H^\dagger}^k$ converges weakly to $(T^\dagger)^{-1}e^{-\beta H_0}T^\dagger = (T^\dagger)^{-1}\rho_0 T^\dagger$ on $\mathcal{D}$. This suggests to define, in analogy with (4.5),

$$\rho = T\rho_0 T^{-1}, \qquad \rho^\dagger = (T^\dagger)^{-1}\rho_0 T^\dagger.$$

Notice now that $(\rho - \mathbb{1})^k = T(\rho_0 - \mathbb{1})^k T^{-1}$, for all $k = 0, 1, 2, \ldots$. Therefore, using the same argument bringing to definitions (4.5), we can check that $\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}(\rho - \mathbb{1})^k$ converges weakly to $T\,\log(\rho_0)T^{-1}$ on $\mathcal{D}$, and $\sum_{k=0}^{n}(-1)^{k-1}\frac{1}{k}(\rho^\dagger - \mathbb{1})^k$ converges weakly to $(T^\dagger)^{-1}\log(\rho_0)T^\dagger$ on $\mathcal{D}$. Hence we put

$$\log\rho := T\,\log(\rho_0)T^{-1}, \qquad \log\rho^\dagger := (T^\dagger)^{-1}\log(\rho_0)T^\dagger,$$

and we define a new von Neumann-like entropy as follows:

$$S_\rho = -\sum_n (\psi_n|\rho\log\rho\varphi_n) = -\sum_n (\rho^\dagger\psi_n|(\log\rho)\varphi_n),$$

under our working assumptions, and in particular the fact that $\rho^\dagger\psi_n \in \mathcal{D}$ and $(\log\rho)\varphi_n \in \mathcal{D}$, we easily conclude that $S_\rho = S_{\rho_0}$.

Remark It is worth pointing out that even in the cases when $H_0\mathcal{D} \subset \mathcal{D}$, we cannot say that $\log(\rho_0) \in \mathcal{L}^\dagger(D)$ as it happens if $\mathcal{D} = D^\infty(H_0)$; due to the assumptions on T ($T\mathcal{D} = \mathcal{D}$; $T^*D = D$) this would imply that also $\log(\rho)$ maps $\mathcal{D}$ into $\mathcal{D}$. Nevertheless in the above computations only the action the $\{\varphi_n\}$'s is involved, were everything goes in the appropriate way.

4.4 KMS-Like Condition

In this section we investigate whether the Gibbs state ω_φ^β satisfies the KMS-condition with respect to $\{\alpha_t^\varphi\}$, that is, for any $X, Y \in \mathcal{L}^\dagger(\mathcal{D})$ there exists a bounded continuous function $f_{X,Y}$ on the strip $\mathcal{S}_\beta := \{z \in \mathbb{C};\ 0 \leqq \operatorname{Im} z \leqq \beta\}$ such that

$$\begin{aligned} f_{X,Y}(t) &= \omega_\varphi^\beta(X\alpha_t^\varphi(Y)), \\ f_{X,Y}(t+\beta i) &= \omega_\varphi^\beta(\alpha_t^\varphi(Y)X), \end{aligned}$$

for all $t \in \mathbb{R}$.

Throughout this section let $\{\varphi_n\}$ be a generalized Riesz system in $\mathcal{H}$ with a constructing pair $(\{f_n\}, T)$ satisfying $TD^\infty(H_0) = D^\infty(H_0)$ and $T^*D^\infty(H_0) = D^\infty(H_0)$ and $\beta > 0$. Here we put $\mathcal{D} := D^\infty(H_0)$. Then, since $e^{-\delta H_0}\mathcal{H} \subset \mathcal{D}$ for any $\delta > 0$, it follows from Lemma 3.1 that

$$Xe^{-\delta H_0} \text{ is trace class for each } \delta > 0 \text{ and } X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}), \tag{4.8}$$

and from Corollarys 3.4 and 4.2.3 that ω_φ^β and ω_ψ^β are faithful states on $\mathcal{L}^\dagger(\mathcal{D})$ and $\{\alpha_t^\varphi\}_{t\in\mathbb{R}}$ and $\{\alpha_t^\psi\}_{t\in\mathbb{R}}$ are quasi-strongly continuous one-parameter groups of automorphisms of $\mathcal{L}^\dagger(\mathcal{D})$. We have the following

Theorem 4.4.1 *For any $X, Y \in \mathcal{L}^\dagger(\mathcal{D})$ there exists a bounded continuous function $f_{X,Y}$ on the strip $\mathcal{S}_\beta$ in $\mathbb{C}$ which is analytic on $0 < \operatorname{Im} z < \beta$ such that*

$$\begin{aligned} f_{X,Y}(t) &= \omega_\varphi^\beta(X\alpha_t^\varphi(Y)), \\ f_{X,Y}(t+\beta i) &= \omega_\varphi^\beta\left((TT^\dagger)^{-1}\alpha_t^\varphi(Y)TT^\dagger X\right), \end{aligned}$$

for all $t \in \mathbb{R}$.

Proof By Theorem 3.2 we have

$$\begin{aligned} \omega_\varphi^\beta(X) &= \frac{1}{Z_\varphi} \operatorname{tr}\left(T^\dagger XTe^{-\beta H_0}\right) \\ &= \frac{1}{Z_\varphi} \operatorname{tr}\left(Te^{-\beta H_0}T^\dagger X\right) \\ &= \frac{1}{Z_\varphi} \operatorname{tr}\left(e^{-\beta H}TT^\dagger X\right), \end{aligned} \tag{4.9}$$

for all $X \in \mathcal{L}^\dagger(\mathcal{D})$. In order to define a function $f_{X,Y}$ on the strip $\mathcal{S}_\beta$ in $\mathbb{C}$, we extend α_t^φ to the strip $\mathcal{S}_\beta$ as follows:

$$\begin{aligned} \alpha_z^\varphi(Y) &= Te^{izH_0}T^{-1}YTe^{-izH_0}T^{-1} \\ &= Te^{-sH_0}e^{itH_0}T^{-1}YTe^{-itH_0}e^{sH_0}T^{-1}, \quad z = t+is \in \mathcal{S}_\beta \text{ and } Y \in \mathcal{L}^\dagger(\mathcal{D}). \end{aligned}$$

Then, $\alpha_z^{\varphi}(Y)$ is not necessarily contained in $\mathcal{L}^{\dagger}(\mathcal{D})$. However, we have

$$\begin{aligned} T^{\dagger}X\alpha_z^{\varphi}(Y)Te^{-\beta H_0} &= T^{\dagger}XTe^{-sH_0}e^{itH_0}T^{-1}YTe^{-itH_0}e^{sH_0}e^{-\beta H_0} \\ &= T^{\dagger}XTe^{-sH_0}\alpha_t^0(T^{-1}YT)e^{-(\beta-s)H_0}. \end{aligned}$$

Hence, because of (4.8), taking into account that $T^{\dagger}XTe^{-sH_0}\alpha_t^0(T^{-1}YT) \in \mathcal{L}^{\dagger}(\mathcal{D})$), we conclude that $T^{\dagger}X\alpha_z^{\varphi}(Y)Te^{-\beta H_0}$ is trace class. Now we put

$$f_{X,Y}(z) := \frac{1}{Z_{\varphi}} \operatorname{tr} \left(T^{\dagger}X\alpha_z^{\varphi}(Y)Te^{-\beta H_0}\right), \quad z \in \mathcal{S}_{\beta}. \tag{4.10}$$

Then $f_{X,Y}(z)$ is analytic on $z \in \mathcal{S}_{\beta}$ with $0 < \operatorname{Im} z < \beta$. Indeed, take arbitrary a sufficient small constant $\delta > 0$ $(0 < \delta < \beta)$. Then we have

$$\begin{aligned} f_{X,Y}(z) &= \frac{1}{Z_{\varphi}} \operatorname{tr} \left(T^{\dagger}XTe^{izH_0}T^{-1}YTe^{-izH_0}T^{-1}Te^{-\beta H_0}\right) \\ &= \frac{1}{Z_{\varphi}} \operatorname{tr} \left(T^{\dagger}XTe^{izH_0}T^{-1}YTe^{-izH_0}T^{-1}Te^{-(\beta-\delta)H_0}e^{-\delta H_0}\right) \\ &= \frac{1}{Z_{\varphi}} \operatorname{tr} \left(e^{-(\beta-\delta)H_0}\left(e^{-\frac{\delta}{2}H_0}T^{\dagger}XT\right)e^{izH_0}\left(T^{-1}YTe^{-\frac{\delta}{2}H_0}\right)e^{-izH_0}\right) \\ &= \frac{1}{Z_{\varphi}} \operatorname{tr} \left(e^{-(\beta-\delta)H_0}\left(e^{-\frac{\delta}{2}H_0}T^{-1}XT\right)\alpha_z^0\left(T^{-1}YTe^{-\frac{\delta}{2}H_0}\right)\right), \\ &= \frac{1}{Z_{\varphi}} \operatorname{tr} \left(e^{-(\beta-\delta)H_0}A\alpha_z^0(B)\right), \end{aligned}$$

where $A := \left(e^{-\frac{\delta}{2}H_0}T^{-1}XT\right)$ and $B := \left(T^{-1}YTe^{-\frac{\delta}{2}H_0}\right)$. By (4.8), A and B are trace class. Hence it is known that

$$z \to \operatorname{tr} \left(e^{-(\beta-\delta)H_0}A\alpha_0^z(B)\right) \quad \text{is analytic on } \mathcal{S}_{\beta-\delta} \text{ with } 0 < \operatorname{Im} z < \beta-\delta \tag{4.11}$$

(see 4.3 in [24]). Then for any $z_0 \in \mathcal{S}_{\beta}$ with $0 < \operatorname{Im} z_0 < \beta$ there exists a constant $\delta > 0$ such that $\operatorname{Im} z_0 < \beta - \delta$. By (4.11), $f_{X,Y}$ is analytic at z_0. Thus $f_{X,Y}$ is analytic on $\mathcal{S}_{\beta}$ with $0 < \operatorname{Im} z < \beta$. Furthermore, by (4.9) we have

$$\begin{aligned} f_{X,Y}(t) &= \frac{1}{Z_{\varphi}} \operatorname{tr} \left(T^{\dagger}X\alpha_t^{\varphi}(Y)Te^{-\beta H_0}\right) \\ &= \frac{1}{Z_{\varphi}} \operatorname{tr} \left(Te^{-\beta H_0}T^{\dagger}X\alpha_t^{\varphi}(Y)\right) \\ &= \omega_{\varphi}^{\beta}\left(X\alpha_t^{\varphi}(Y)\right) \end{aligned}$$

and

$$\begin{aligned}
& f_{X,Y}(t+\beta i) \\
&= \frac{1}{Z_\varphi} \operatorname{tr} \left(T e^{-\beta H_0} T^{-1} T T^\dagger X \left(T e^{-\beta H_0} T^{-1} \right) \right. \\
&\quad \left. \left(T e^{itH_0} \right) T^{-1} Y \left(T e^{-itH_0} T^{-1} \right) \left(T e^{-\beta H_0} T^{-1} \right) \right) \\
&= \frac{1}{Z_\varphi} \operatorname{tr} \left(T T^\dagger X \left(T e^{-\beta H_0} T^{-1} \right) T \alpha_t^0 \left(T^{-1} Y T \right) T^{-1} \right) \\
&= \frac{1}{Z_\varphi} \operatorname{tr} \left(e^{-\beta H} T \alpha_t^0 (T^{-1} Y T) T^\dagger X \right) \\
&= \frac{1}{Z_\varphi} \operatorname{tr} \left(e^{-\beta H} \alpha_t^\varphi (Y) T T^\dagger X \right) \\
&= \frac{1}{Z_\varphi} \operatorname{tr} \left(e^{-\beta H} T T^\dagger (T T^\dagger)^{-1} \alpha_t^\varphi (Y) T T^\dagger X \right) \\
&= \omega_\varphi^\beta \left((T T^\dagger)^{-1} \alpha_t^\varphi (Y) T T^\dagger X \right).
\end{aligned}$$

This completes the proof. □

Thus ω_φ^β does not satisfy the KMS-condition with respect to $\{\alpha_t^\varphi\}$, but still it satisfies the KMS-like condition with respect to $\{\alpha_t^\varphi\}$, as Theorem 4.4.1 shows. Furthermore, we have a similar result for the Gibbs state ω_ψ^β as follows:

Theorem 4.4.2 *For any $X, Y \in \mathcal{L}^\dagger(\mathcal{D})$ there exists a bounded continuous function $F_{X,Y}$ on the strip $\mathcal{S}_\beta$ in $\mathbb{C}$ which is analytic on $0 < \operatorname{Im} z < \beta$ such that*

$$\begin{aligned}
F_{X,Y}(t) &= \omega_\psi^\beta (X \alpha_t^\psi (Y)), \\
F_{X,Y}(t+\beta i) &= \omega_\psi^\beta \left((T T^\dagger) \alpha_t^\psi (Y) (T T^\dagger)^{-1} X \right),
\end{aligned}$$

for all $t \in \mathbb{R}$.

Remark We do not know whether Theorems 4.4.1 and 4.4.2 hold for a general subspace $\mathcal{D}$ satisfying Assumption 3. This is because we do not know whether (4.10) holds for unbounded operators $T^\dagger X \alpha_z^\varphi (Y) T$.

5 Gibbs States and Unbounded Tomita–Takesaki Theory

5.1 Unbounded Tomita–Takesaki Theory in Hilbert Space of Hilbert–Schmidt Operators

In this subsection we review the basic definitions and results of unbounded Tomita–Takesaki theory in the Hilbert space of Hilbert–Schmidt operators. For details refer to [25]. Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{H} \otimes \bar{\mathcal{H}}$ be the Hilbert space of all Hilbert–Schmidt operators on $\mathcal{H}$ with the inner product

$$(S|T) := \operatorname{tr}(T^* S), \quad S, T \in \mathcal{H} \otimes \bar{\mathcal{H}}.$$

Let $\mathcal{D}$ be a dense subspace in $\mathcal{H}$ such that $\mathcal{L}^\dagger(\mathcal{D})$ is closed, namely $\mathcal{D} = \cap_{X \in \mathcal{L}^\dagger(\mathcal{D})} D(\bar{X})$. We define a dense subspace $\sigma_2(\mathcal{D})$ of $\mathcal{H} \otimes \bar{\mathcal{H}}$ by

$$\sigma_2(\mathcal{D}) := \{T \in \mathcal{H} \otimes \bar{\mathcal{H}};\ T\mathcal{H} \subset \mathcal{D} \text{ and } XT \in \mathcal{H} \otimes \bar{\mathcal{H}} \text{ for all } X \in \mathcal{L}^\dagger(\mathcal{D})\}$$

and an operator $\pi(X)$ on $\sigma_2(\mathcal{D})$ by

$$\pi(X) := XT, \quad X \in \mathcal{L}^\dagger(\mathcal{D}),\ T \in \sigma_2(\mathcal{D}).$$

Then π is a $*$-homomorphism of the O^*-algebra $\mathcal{L}^\dagger(\mathcal{D})$ into the O^*-algebra $\mathcal{L}^\dagger(\sigma_2(\mathcal{D}))$, and hence $\pi(\mathcal{L}^\dagger(\mathcal{D}))$ is an O^*-algebra on $\sigma_2(\mathcal{D})$ in $\mathcal{H} \otimes \bar{\mathcal{H}}$. We can also define a bounded $*$-homomorphism π'' and an anti $*$-homomorphism π' of $B(\mathcal{H})$ into the C^*-algebra $B(\mathcal{H} \otimes \bar{\mathcal{H}})$ by

$$\pi''(A)T = AT \quad \text{and} \quad \pi'(A)T = TA, \quad A \in B(\mathcal{H}),\ T \in \mathcal{H} \otimes \bar{\mathcal{H}},$$

and $\pi''(B(\mathcal{H}))$ and $\pi'(B(\mathcal{H}))$ are von Neumann algebras on $\mathcal{H} \otimes \bar{\mathcal{H}}$ satisfying $\pi'(B(\mathcal{H})) = \pi''(B(\mathcal{H}))' = J\pi''(B(\mathcal{H}))J$, where $JT = T^*$ for any $T \in \mathcal{H} \otimes \bar{\mathcal{H}}$. Then it follows from Lemma 2.4.14 in [25] that

$$\pi(\mathcal{L}^\dagger(\mathcal{D}))'_w = \pi'(B(\mathcal{H})) \text{ and } \left(\pi(\mathcal{L}^\dagger(\mathcal{D}))'_w\right)' = \pi''(B(\mathcal{H})). \tag{5.1}$$

Suppose that Ω is a non-singular positive self-adjoint operator on $\mathcal{H}$ belonging to $\sigma_2(\mathcal{D})$. Then, it follows from Lemma 2.4.16 in [25] that Ω is a strongly cyclic vector for the O^*-algebra $\pi(\mathcal{L}^\dagger(\mathcal{D}))$ (namely, $\pi(\mathcal{L}^\dagger(\mathcal{D}))\Omega$ is $t_{\pi(\mathcal{L}^\dagger(\mathcal{D}))}$-dense in $\mathcal{H} \otimes \bar{\mathcal{H}}$) and $\pi(\mathcal{L}^\dagger(\mathcal{D}))'_w\Omega$ is dense in $\mathcal{H} \otimes \bar{\mathcal{H}}$, and hence it is a cyclic and separating vector for the von Neumann algebra $\pi''(B(\mathcal{H}))$, which implies that $\pi''(B(\mathcal{H}))\Omega$ is a left Hilbert algebra in $\mathcal{H} \otimes \bar{\mathcal{H}}$ under the following multiplication and involution:

$$\begin{aligned} \left(\pi''(A)\Omega\right)\left(\pi''(B)\Omega\right) &:= \pi''(AB)\Omega, \\ \left(\pi''(A)\Omega\right)^\sharp &:= \pi''(A^*)\Omega, \quad A, B \in B(\mathcal{H}). \end{aligned}$$

Let $S''_{\mathfrak{A}} = J''_\Omega \triangle_\Omega''^{\frac{1}{2}}$ be the polar decomposition of the conjugate linear closed operator S''_Ω which is the closure of the involution $\pi''(A)\Omega \to \pi''(A^*)\Omega$. Then J''_Ω is a conjugate linear isometry on $\mathcal{H} \otimes \bar{\mathcal{H}}$ and $\triangle''_\Omega$ is a non-singular positive self-adjoint operator in $\mathcal{H} \otimes \bar{\mathcal{H}}$ and they are called the *modular conjugation* and the *modular operator* of the left Hilbert algebra $\pi''(B(\mathcal{H}))\Omega$. By the Tomita theorem a strongly continuous one-parameter group $\{(\delta_t^\Omega)''\}_{t\in\mathbb{R}}$ of the von Neumann algebra $\pi''(B(\mathcal{H}))$ is defined by

$$(\delta_t^\Omega)''(\pi''(A)) = \triangle_\Omega''^{\,it}\pi''(A)\triangle_\Omega''^{\,-it}, \quad A \in B(\mathcal{H}),\ t \in \mathbb{R},$$

and it is called the *modular automorphism group* of $\pi''(B(\mathcal{H}))$. For the Tomita–Takesaki theory we refer to [26]. Then it follows from Theorem 2.4.18 in [25] that

$$J_\Omega'' = J \quad \text{and} \quad \triangle_\Omega'' = \pi'(\Omega^{-2})\pi''(\Omega^2), \tag{5.2}$$

where the positive self-adjoint operator $\pi'(\Omega^{-2})$ is defined by

$$\begin{cases} D(\pi'(\Omega^{-2})) = \{T \in \mathcal{H}\otimes\bar{\mathcal{H}};\, T\Omega^{-2} \text{ is closable and } \overline{T\Omega^{-2}} \in \mathcal{H}\otimes\bar{\mathcal{H}}\} \\ \qquad\qquad \pi'(\Omega^{-2})T = \overline{T\Omega^{-2}},\, T \in D(\pi'(\Omega^{-2})). \end{cases}$$

By (5.1) we have

$$\pi(\mathcal{L}^\dagger(\mathcal{D}))'_w\Omega = \pi'(B(\mathcal{H}))\Omega,$$
$$(\pi(\mathcal{L}^\dagger(\mathcal{D}))'_w)'\Omega = \pi''(B(\mathcal{H}))\Omega,$$

and so the involution: $\pi(X)\Omega \to \pi(X^\dagger)\Omega$, $X \in \mathcal{L}^\dagger(\mathcal{D})$ is a conjugate linear closable operator in $\mathcal{H}\otimes\bar{\mathcal{H}}$ and its closure is denoted by $S_\mathfrak{A}$. Let $S_\mathfrak{A} = J_\mathfrak{A}\triangle_\mathfrak{A}^{\frac{1}{2}}$ be the polar decomposition of $S_\mathfrak{A}$. Then we can show that $S_\mathfrak{A} = S_\mathfrak{A}''$, and so $J_\mathfrak{A} = J_\mathfrak{A}''$ and $\triangle_\mathfrak{A} = \triangle_\mathfrak{A}''$. Hereafter, we use $S_\mathfrak{A}$, $J_\mathfrak{A}$, $\triangle_\mathfrak{A}$ and $\{\delta_t^\Omega\}_{t\in\mathbb{R}}$. Suppose that $\Omega^{it}\mathcal{D} \subset \mathcal{D}$, for all $t \in \mathbb{R}$, namely $\Omega^{it} \in \mathcal{L}^\dagger(\mathcal{D})$. Then since

$$\triangle_\Omega^{it} = \pi'(\Omega^{-2it})\pi''(\Omega^{2it}) = \pi'(\Omega^{-2it})\pi(\Omega^{2it}) \in \mathcal{L}^\dagger(\mathcal{D}), \quad t \in \mathbb{R}$$

by (5.2), we can define a one-parameter group $\{\sigma_t^\Omega\}_{t\in\mathbb{R}}$ of the O^*-algebra $\pi(\mathcal{L}^\dagger(\mathcal{D}))$ by

$$\sigma_t^\Omega(\pi(X)) := \triangle_\Omega^{it}\pi(X)\triangle_\Omega^{-it}, \quad X \in \mathcal{L}^\dagger(\mathcal{D}),\ t \in \mathbb{R},$$

and we see that

$$\begin{aligned} \sigma_t^\Omega(\pi(X)) &= \pi'(\Omega^{-2it})\pi(\Omega^{2it})\pi(X)\pi'(\Omega^{2it})\pi(\Omega^{-2it}) \\ &= \pi(\Omega^{2it})\pi(X)\pi(\Omega^{-2it}) \\ &= \pi(\Omega^{2it}X\Omega^{-2it}), \end{aligned}$$

for all $X \in \mathcal{L}^\dagger(\mathcal{D})$ and $t \in \mathbb{R}$. This $\{\sigma_t^\Omega\}$ is called the *modular automorphism group* of $\pi(\mathcal{L}^\dagger(\mathcal{D}))$. Thus we have the following

Proposition 5.1.1 *Suppose that Ω is a non-singular positive self-adjoint operator on $\mathcal{H}$ belonging to $\sigma_2(\mathcal{D})$ and $\Omega^{it}\mathcal{D} \subset \mathcal{D}$, for all $t \in \mathbb{R}$. Then*

$$\sigma_t^{\Omega}(X) := \Omega^{it} X \Omega^{-it}, \quad X \in \mathcal{L}^{\dagger}(\mathcal{D}), \ t \in \mathbb{R}$$

is a one-parameter group of $*$-automorphisms of $\mathcal{L}^{\dagger}(\mathcal{D})$, which is induced by the modular automorphism group $\{\sigma_t^{\Omega}\}_{t\in\mathbb{R}}$ of $\pi(\mathcal{L}^{\dagger}(\mathcal{D}))$.

5.2 Modular Automorphism Group Defined by the Gibbs State ω_{φ}^{β}

Let $\{\varphi_n\}$ be a generalized Riesz system with a constructing pair $(\{f_n\}, T)$ and H_0 be a standard Hamiltonian. We assume the following

Assumption 4 There exists a dense subspace $\mathcal{D}$ in $\mathcal{H}$ such that

(i) $e^{-\frac{\beta}{2}H_0}\mathcal{H} \subset \mathcal{D} \subset D(T) \cap D(T^*)$,
(ii) $T\lceil_{\mathcal{D}} \in \mathcal{L}(\mathcal{D})$,
(iii) $\mathcal{L}^{\dagger}(\mathcal{D})$ is self-adjoint, namely $\mathcal{D} = \cap_{X\in\mathcal{L}^{\dagger}(\mathcal{D})} D(X^*)$.

As seen in Sect. 3, the Gibbs state ω_0 on $\mathcal{L}^{\dagger}(\mathcal{D})$ is defined by

$$\omega_0(X) = \frac{1}{Z_0} \operatorname{tr} \left(e^{-\frac{\beta}{2}H_0} X e^{-\frac{\beta}{2}H_0} \right), \quad X \in \mathcal{L}^{\dagger}(\mathcal{D}).$$

Then we see that

$$\Omega_0 := \frac{1}{\sqrt{Z_0}} e^{-\frac{\beta}{2}H_0} \in \left(\sigma_2(\mathcal{L}^{\dagger}(\mathcal{D})) \right)_+$$

and

$$\omega_0(X) = (\pi(X)\Omega_0|\Omega_0), \quad X \in \mathcal{L}^{\dagger}(\mathcal{D}).$$

By Proposition 5.1.1, the modular automorphism group $\{\sigma_t^{\Omega_0}\}_{t\in\mathbb{R}}$ of $\pi(\mathcal{L}^{\dagger}(\mathcal{D}))$ defined by ω_0 coincides with $\{\alpha_t^0\}_{t\in\mathbb{R}}$. By Theorem 3.2, the Gibbs state ω_{φ}^{β} for $\{\varphi_n\}$ is defined by

$$\omega_{\varphi}^{\beta}(X) = \frac{1}{Z_{\varphi}} \operatorname{tr} \left(\left(Te^{-\frac{\beta}{2}H_0} \right)^* X \left(Te^{-\frac{\beta}{2}H_0} \right) \right),$$

for all $X \in \mathcal{L}^{\dagger}(\mathcal{D})$. Here we shall extend results in Sect. 3 for the Gibbs state ω_{φ}^{β} on $\mathcal{L}^{\dagger}(\mathcal{D})$.

Let $(Te^{-\frac{\beta}{2}H_0})^* = U|(Te^{-\frac{\beta}{2}H_0})^*|$ be the polar decomposition of $(Te^{-\frac{\beta}{2}H_0})^*$. By Lemma 3.1 and Assumption 4, (i) and (ii), we have

$$\left(Te^{-\frac{\beta}{2}H_0} \right) \left(Te^{-\frac{\beta}{2}H_0} \right)^* \mathcal{H} \subset T \left(e^{-\frac{\beta}{2}H_0} \left(Te^{-\frac{\beta}{2}H_0} \right)^* \right) \mathcal{H} \subset T\mathcal{D} \subset \mathcal{D}$$

and since $\mathcal{L}^\dagger(\mathcal{D})$ is a self-adjoint O^*-algebra on $\mathcal{D}$, it follows by Lemma 2.4 in [27] that

$$\left|\left(Te^{-\frac{\beta}{2}H_0}\right)^*\right|\mathcal{H} = \left(\left(Te^{-\frac{\beta}{2}H_0}\right)\left(Te^{-\frac{\beta}{2}H_0}\right)^*\right)^{\frac{1}{2}}\mathcal{H} \subset \mathcal{D}. \tag{5.3}$$

From the above, we put

$$\Omega_\varphi := \frac{1}{\sqrt{Z_\varphi}}\left|(Te^{-\frac{\beta}{2}H_0})^*\right|.$$

Since $XT \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ for all $X \in \mathcal{L}^\dagger(\mathcal{D})$ by Assumption 4, (i), it follows from Lemma 3.1 that

$$\begin{aligned} X\Omega_\varphi &= \frac{1}{\sqrt{Z_\varphi}}X\left|(Te^{-\frac{\beta}{2}H_0})^*\right| \\ &= \frac{1}{\sqrt{Z_\varphi}}XTe^{-\frac{\beta}{2}H_0}U \in \mathcal{H}\otimes\bar{\mathcal{H}}, \end{aligned}$$

for all $X \in \mathcal{L}^\dagger(\mathcal{D})$, which implies by (5.3) that $\Omega_\varphi \in \sigma_2(\mathcal{D})$. Thus, Ω_φ is a non-singular positive self-adjoint Hilbert Schmidt operator on $\mathcal{H}$ contained in $\sigma_2(\mathcal{D})$. Hence, Ω_φ is a strongly cyclic and separating vector for $\pi(\mathcal{L}^\dagger(\mathcal{D}))$ and

$$\begin{aligned} \omega_\varphi^\beta(X) &= \operatorname{tr}\left(U\Omega_\varphi X\Omega_\varphi U^*\right) \\ &= \operatorname{tr}\left(\Omega_\varphi X\Omega_\varphi\right) \\ &= (\pi(X)\Omega_\varphi|\Omega_\varphi), \quad X \in \mathcal{L}^\dagger(\mathcal{D}). \end{aligned}$$

Remark The previous expression for ω_φ is, of course, the GNS representation (up to unitary equivalences) and the cyclic and separating vector Ω_φ which is actually an operator in $\sigma_2(\mathcal{D})$ helps with identifying the *density operator* ρ for which one can write $\omega_\varphi^\beta(X) = \operatorname{tr}(X\rho)$.

By Proposition 5.1.1, we have the following

Theorem 5.2.1 *Suppose that $\{\varphi_n\}$ be a generalized Riesz system with a constructing pair $(\{f_n\}, T)$ and there exists a dense subspace $\mathcal{D}$ in $\mathcal{H}$ satisfying Assumption 4. Then $\Omega_\varphi := \frac{1}{\sqrt{Z_\varphi}}|(Te^{-\frac{\beta}{2}H_0})^*|$ is a non-singular strongly cyclic and separating vector for $\pi(\mathcal{L}^\dagger(\mathcal{D}))$ contained in $\sigma_2(\mathcal{D})$ and the Gibbs state ω_φ^β is represented as*

$$\omega_\varphi^\beta(X) = (\pi(X)\Omega_\varphi|\Omega_\varphi), \quad X \in \mathcal{L}^\dagger(\mathcal{D}).$$

Furthermore, if $\Omega_\varphi^{it}\mathcal{D} \subset \mathcal{D}$ for all $t \in \mathbb{R}$, then $\sigma_t^{\Omega_\varphi} := \Omega_\varphi^{it}X\Omega_\varphi^{it}$, $X \in \mathcal{L}^\dagger(\mathcal{D})$, $t \in \mathbb{R}$ is a one-parameter group of $*$-automorphisms of $\mathcal{L}^\dagger(\mathcal{D})$, which is induced by the modular automorphism group

$$\sigma_t^{\Omega_\varphi}(\pi(X)) := \triangle_{\Omega_\varphi}^{it}\pi(X)\triangle_{\Omega_\varphi}^{-it}, \quad t \in \mathbb{R}$$

of $\pi(\mathcal{L}^\dagger(\mathcal{D}))$.

Remark If $\bar{T}$ commutes to e^{-H_0}, that is $e^{-H_0}\bar{T} \subset \bar{T}e^{-H_0}$, then $\alpha_t^\varphi(X) = |T^*|^{it}\sigma_{2t}^\varphi(X)|T^*|^{-it}$, for all $X \in \mathcal{L}^\dagger(\mathcal{D})$ and $t \in \mathbb{R}$. Since σ_t^φ is a $*$-automorphism of $\mathcal{L}^\dagger(\mathcal{D})$, but α_t^φ is not a $*$-automorphism, these two one-parameter groups $\{\sigma_t^\varphi\}$ and $\{\alpha_t^\varphi\}$ of automorphisms of $\mathcal{L}^\dagger(\mathcal{D})$ have no relation in general.

For the Gibbs state ω_ψ^β on $\mathcal{L}^\dagger(\mathcal{D})$ we similarly have the following

Theorem 5.2.2 *Let* $\{\varphi_n\}$ *be a generalized Riesz system with a constructing pair* $(\{f_n\}, T)$, $n \in \mathbb{N}_0$. *Suppose that there exists a dense subspace* $\mathcal{D}$ *in* $\mathcal{H}$ *satisfying*

(i) $e^{-\frac{\beta}{2}H_0}\mathcal{H} \subset \mathcal{D} \subset D(T^{-1}) \cap D((T^{-1})^*)$,
(ii) $(T^{-1})^*\lceil_{\mathcal{D}} \in \mathcal{L}(\mathcal{D})$,
(iii) $\mathcal{L}^\dagger(\mathcal{D})$ *is self-adjoint.*

Then $\Omega_\psi := \frac{1}{\sqrt{Z_\psi}}\left|\left((T^{-1})^*e^{-\frac{\beta}{2}H_0}\right)^*\right|$ *is a non-singular strongly cyclic and separating vector for* $\pi(\mathcal{L}^\dagger(\mathcal{D}))$ *contained in* $\sigma_2(\mathcal{D})$ *and the Gibbs state* ω_ψ^β *is represented as*

$$\omega_\psi^\beta(X) = (\pi(X)\Omega_\psi|\Omega_\psi), \quad X \in \mathcal{L}^\dagger(\mathcal{D}).$$

Furthermore, if $\Omega_\psi^{it}\mathcal{D} \subset \mathcal{D}$ *for all* $t \in \mathbb{R}$, *then*

$$\sigma_t^{\Omega_\psi}(X) := \Omega_\psi^{it}X\Omega_\psi^{-it}, \quad X \in \mathcal{L}^\dagger(\mathcal{D}),\ t \in \mathbb{R}$$

is a one-parameter group of $*$*-automorphisms of* $\mathcal{L}^\dagger(\mathcal{D})$, *which is induced by the modular automorphism group*

$$\sigma_t^{\Omega_\psi}(\pi(X)) := \triangle_{\Omega_\psi}^{it}\pi(X)\triangle_{\Omega_\psi}^{-it}, \quad t \in \mathbb{R}$$

of $\pi(\mathcal{L}^\dagger(\mathcal{D}))$.

6 Conclusions

In this paper we have discussed how to generalize the standard notions of Heisenberg dynamics, Gibbs states, KMS- condition and Tomita–Takesaki theory to the case in which the dynamics is driven by a non self-adjoint Hamiltonian, as it often happens in PT- and in pseudo-hermitian quantum mechanics and we have chosen to consider observables as elements of $\mathcal{L}^\dagger(\mathcal{D})$. We have also seen how generalized Riesz systems can be used in this context, and how the results deduced here differ from the standard ones. We have also discussed some preliminary results on entropy and on the Tomita–Takesaki theory in our settings.

Of course, many other aspects could be considered in future, from the use of Gibbs states defined by generalized Riesz systems in the analysis of concrete physical systems to more mathematical aspects. For instance, since it is often difficult or even impossible to find a common invariant dense domain $\mathcal{D}$ for the observables, one could try to enlarge the setting to some other relevant subset of $\mathcal{L}^{\dagger}(\mathcal{D}, \mathcal{H})$. We plan to work on these and other aspects of our framework soon.

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